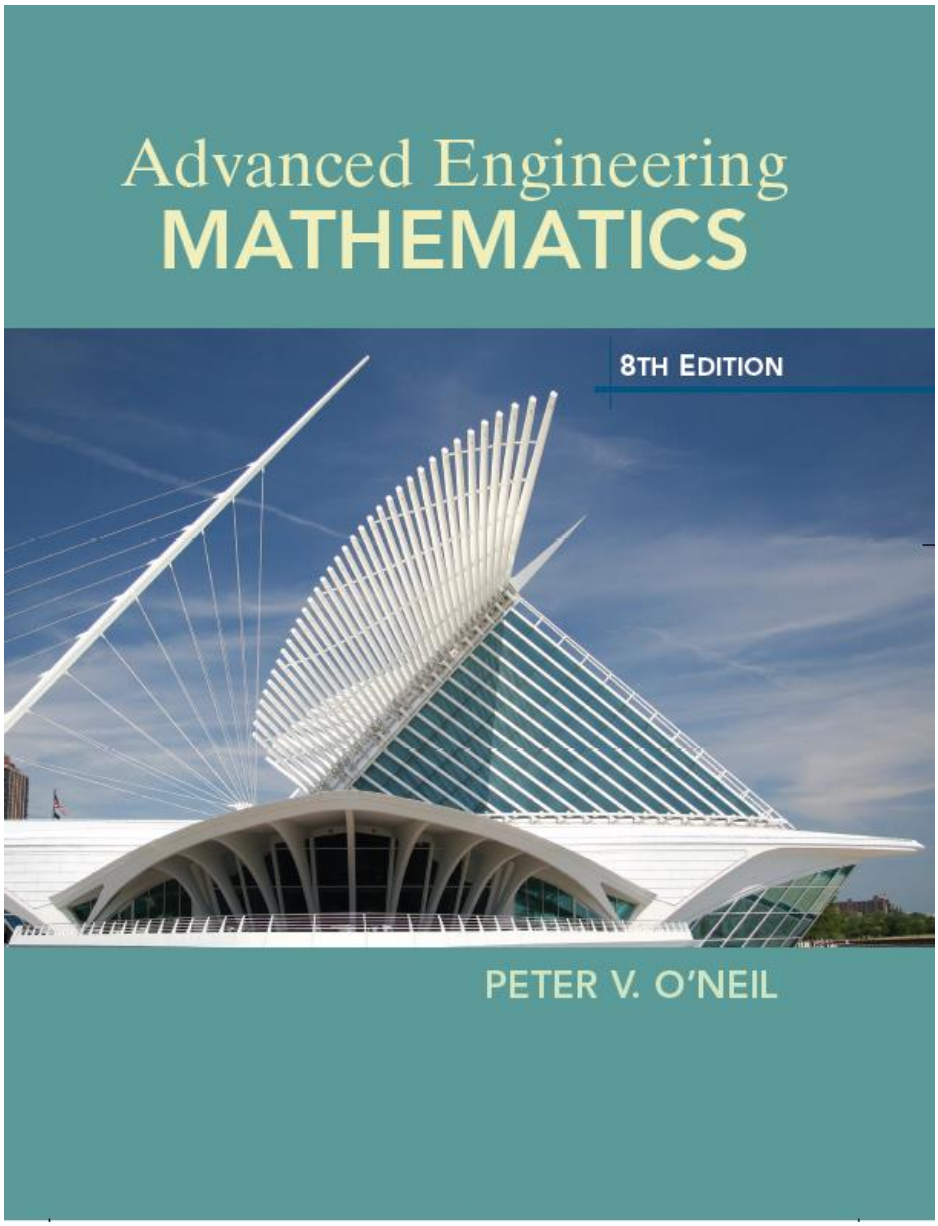
An Instructor's Solutions Manual to Accompany

ADVANCED ENGINEERING MATHEMATICS, 8TH EDITION

PETER V. O'NEIL



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INSTRUCTOR'S SOLUTIONS MANUAL

TO ACCOMPANY

ADVANCED ENGINEERNG MATHEMATICS

8th EDITION

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Contents

Chapter 1

First-Order Differential Equations

1.1 Terminology and Separable Equations

1. The differential equation is separable because it can be written

$$3y^2 \frac{dy}{dx} = 4x,$$

or, in differential form,

$$3y^2 \, dy = 4x \, dx.$$

Integrate to obtain

$$y^3 = 2x^2 + k.$$

This implicitly defines a general solution, which can be written explicitly as

$$y = (2x^2 + k)^{1/3},$$

with k an arbitrary constant.

2. Write the differential equation as

$$x \frac{dy}{dx} = -y,$$

which separates as

$$\frac{1}{y} \, dy = -\frac{1}{x} \, dx$$

if $x \neq 0$ and $y \neq 0$. Integrate to get

$$\ln |y| = -\ln |x| + k.$$

Then $\ln |xy| = k$, so

$$xy = c$$

with c constant ($c = e^k$). $y = 0$ is a singular solution, satisfying the original differential equation.

3. If $\cos(y) \neq 0$, the differential equation is

$$\begin{aligned}\frac{y}{dx} &= \frac{\sin(x+y)}{\cos(y)}\\ &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(y)}\\ &= \sin(x) + \cos(x)\tan(y).\end{aligned}$$

There is no way to separate the variables in this equation, so the differential equation is not separable.

4. Write the differential equation as

$$e^x e^y \frac{dy}{dx} = 3x,$$

which separates in differential form as

$$e^y\,dy = 3xe^{-x}\,dx.$$

Integrate to get

$$e^y = -3e^{-x}(x+1) + c,$$

with c constant. This implicitly defines a general solution.

5. The differential equation can be written

$$x\frac{dy}{dx} = y^2 - y,$$

or

$$\frac{1}{y(y-1)}\,dy = \frac{1}{x}\,dx,$$

and is therefore separable. Separating the variables assumes that $y \neq 0$ and $y \neq 1$. We can further write

$$\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \frac{1}{x}\,dx.$$

Integrate to obtain

$$\ln|y-1| - \ln|y| = \ln|x| + k.$$

Using properties of the logarithm, this is

$$\ln\left|\frac{y-1}{xy}\right| = k.$$

Then

$$\frac{y-1}{xy} = c,$$

with $c = e^k$ constant. Solve this for y to obtain the general solution

$$y = \frac{1}{1-cx}.$$

$y = 0$ and $y = 1$ are singular solutions because these satisfy the differential equation, but were excluded in the algebra of separating the variables.

6. The differential equation is not separable.

7. The equation is separable because it can be written in differential form as

$$\frac{\sin(y)}{\cos(y)}\,dy = \frac{1}{x}\,dx.$$

This assumes that $x \neq 0$ and $\cos(y) \neq 0$. Integrate this equation to obtain

$$-\ln|\cos(y)| = \ln|x| + k.$$

This implicitly defines a general solution. From this we can also write

$$\sec(y) = cx$$

with c constant.

The algebra of separating the variables required that $\cos(y) \neq 0$. Now $\cos(y) = 0$ if $y = (2n+1)\pi/2$, with n any integer. Now $y = (2n+1)\pi/2$ also satisfies the original differential equation, so these are singular solutions.

8. The differential equation itself requires that $y \neq 0$ and $x \neq -1$. Write the equation as

$$\frac{x}{y}\frac{dy}{dx} = \frac{2y^2+1}{x}$$

and separate the variables to get

$$\frac{1}{y(2y^2+1)}\,dy = \frac{1}{x(x+1)}\,dx.$$

Use a partial fractions decomposition to write this as

$$\left(\frac{1}{y} - \frac{2y}{2y^2+1}\right)dy = \left(\frac{1}{x} - \frac{1}{x+1}\right)dx.$$

Integrate to obtain

$$\ln|y| - \frac{1}{2}\ln(1+2y^2) = \ln|x| - \ln|x+1| + c$$

with c constant. This implicitly defines a general solution. We can go a step further by writing this equation as

$$\ln\left(\frac{y}{\sqrt{1+2y^2}}\right) = \ln\left(\frac{x}{x+1}\right) + c$$

and take the exponential of both sides to get

$$\frac{y}{\sqrt{1+2y^2}} = k\left(\frac{x}{x+1}\right),$$

which also defines a general solution.

9. The differential equation is

$$\frac{dy}{dx} = e^x - y + \sin(y),$$

and this is not separable. It is not possible to separate all terms involving x on one side of the equation and all terms involving y on the other.

10. Substitute

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y),$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
and
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

into the differential equation to get the separated differential form

$$(\cos(y) - \sin(y))\,dy = (\cos(x) - \sin(x))\,dx.$$

Integrate to obtain the implicitly defined general solution

$$\cos(y) + \sin(y) = \cos(x) + \sin(x) + c.$$

11. If $y \neq -1$ and $x \neq 0$, we obtain the separated equation

$$\frac{y^2}{y+1}\,dy = \frac{1}{x}\,dx.$$

To make the integration easier, write this as

$$\left(y - 1 + \frac{1}{1+y}\right)dy = \frac{1}{x}\,dx.$$

Integrate to obtain

$$\frac{1}{2}y^2 - y + \ln|1+y| = \ln|x| + c.$$

This implicitly defines a general solution. The initial condition is $y(3e^2) = 2$, so put $y = 2$ and $x = 3e^2$ to obtain

$$2 - 2 + \ln(3) = \ln(3e^2) + c.$$

Now

$$\ln(3e^2) = \ln(3) + \ln(e^2) = \ln(3) + 2,$$

so

$$\ln(3) = \ln(3) + 2 + c.$$

Then $c = -2$ and the solution of the initial value problem is implicitly defined by

$$\frac{1}{2}y^2 - y + \ln|1 + y| = \ln|x| - 2.$$

12. Integrate

$$\frac{1}{y+2}\,dy = 3x^2\,dx,$$

assuming that $y \neq -2$, to obtain

$$\ln|2 + y| = x^3 + c.$$

This implicitly defines a general solution. To have $y(2) = 8$, let $x = 2$ and $y = 8$ to obtain

$$\ln(10) = 8 + c.$$

The solution of the initial value problem is implicitly defined by

$$\ln|2 + y| = x^3 + \ln(10) - 8.$$

We can take this a step further and write

$$\ln\left(\frac{2+y}{10}\right) = x^3 - 8.$$

By taking the exponential of both sides of this equation we obtain the explicit solution

$$y = 10e^{x^3-8} - 2.$$

13. With $\ln(y^x) = x\ln(y)$, we obtain the separated equation

$$\frac{\ln(y)}{y}\,dy = 3x\,dx.$$

Integrate to obtain

$$(\ln(y))^2 = 3x^2 + c.$$

For $y(2) = e^3$, we need

$$(\ln(e^3))^2 = 3(4) + c,$$

or $9 = 12 + c$. Then $c = -3$ and the solution of the initial value problem is defined by

$$(\ln(y))^2 = 3x^2 - 3.$$

Solve this to obtain the explicit solution

$$y = e^{\sqrt{3(x^2-1)}}$$

if $|x| > 1$.

14. Because $e^{x-y^2} = e^x e^{-y^2}$, the variables can be separated to obtain

$$2ye^{y^2}\,dy = e^x\,dx.$$

Integrate to get

$$e^{y^2} = e^x + c.$$

To satisfy $y(4) = -2$ we need

$$e^4 = e^4 + c$$

so $c = 0$ and the solution of the initial value problem is implicitly defined by

$$e^{y^2} = e^x,$$

which reduces to the simpler equation

$$x = y^2.$$

Because $y(4) = -2$, the explicit solution is $y = -\sqrt{x}$ for $x > 0$.

15. Separate the variables to obtain

$$y\cos(3y)\,dy = 2x\,dx.$$

Integrate to get

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 + c,$$

which implicitly defines a general solution. For $y(2/3) = \pi/3$, we need

$$\frac{1}{3}\frac{\pi}{3}\sin(\pi) + \frac{1}{9}\cos(\pi) = \frac{4}{9} + c.$$

This reduces to

$$-\frac{1}{9} = \frac{4}{9} + c,$$

so $c = -5/9$ and the solution of the initial value problem is implicitly defined by

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 - \frac{5}{9},$$

or

$$3y\sin(3y) + \cos(3y) = 9x^2 - 1.$$

16. Let $T(t)$ be the temperature function. By Newton's law of cooling, $T'(t) = k(T-60)$ for some constant k to be determined. This equation is separable and is easily solved to obtain:

$$T(t) = 60 + 30e^{kt}.$$

To determine k, use the fact that $T(10) = 88$:

$$T(10) = 60 + 30e^{10k} = 88.$$

Then

$$e^{10k} = \frac{88 - 60}{30} = \frac{14}{15},$$

so

$$k = \frac{1}{10}\ln(14/15).$$

Now we know the temperature function completely:

$$\begin{aligned} T(t) = 60 + 30e^{kt} &= 60 + 30\left(e^{10k}\right)^{t/10} \\ &= 60 + 30\left(\frac{14}{15}\right)^{t/10}. \end{aligned}$$

We want to know $T(20)$, so compute

$$T(20) = 60 + 30\left(\frac{14}{15}\right)^{2} \approx 86.13$$

degrees Fahrenheit. To see how long it will take for the object to reach 65 degrees, solve for t in

$$T(t) = 65 = 60 + 30\left(\frac{14}{15}\right)^{t/10}.$$

Then

$$\left(\frac{14}{15}\right)^{t/10} = \frac{65 - 60}{30} = \frac{1}{6},$$

so

$$\frac{t}{10}\ln(14/15) = \ln(1/6) = -\ln(6).$$

The object reaches 65 degrees at time

$$t = -\frac{10\ln(6)}{\ln(14/15)} \approx 259.7$$

minutes.

17. Suppose the thermometer was removed from the house at time $t = 0$, and let $T(t)$ be the temperature function. Let A be the ambient temperature outside the house (assumed constant). By Newton's law,

$$T'(t) = k(t - A).$$

We are also given that $T(0) = 70$ and $T(5) = 60$. Further, fifteen minutes after being removed from the house, the thermometer reads 50.4, so

$$T(15) = 50.4.$$

We want to determine A, the constant outside temperature. From the differential equation for T,

$$\frac{1}{T - A}\, dT = kdt.$$

Integrate this, as we have done before, to get

$$T(t) = A + ce^{kt}.$$

Now,

$$T(0) = 70 = A + c,$$

so $c = 70 - A$ and

$$T(t) = A + (70 - A)e^{kt}.$$

Now use the other two conditions:

$$T(5) = A + (70 - A)e^{5k} = 15.5 \text{ and } T(15) = A + (70 - A)e^{15k} = 50.4.$$

From the equation for $T(5)$, solve for e^{5k} to get

$$e^{5k} = \frac{60 - A}{70 - A}.$$

Then

$$e^{15k} = \left(e^{5k}\right)^3 = \left(\frac{60 - A}{70 - A}\right)^3.$$

Substitute this into the equation $T(15)$ to get

$$(70 - A)\left(\frac{60 - A}{70 - A}\right)^3 = 50.4 - A.$$

Then

$$(60 - A)^3 = (50.4 - A)(70 - A)^2.$$

The cubic terms cancel and this reduces to the quadratic equation

$$10.4A^2 - 1156A + 30960 = 0,$$

with roots 45 and (approximately) 66.15385. Clearly the outside temperature must be less than 50, and must therefore equal 45 degree.

18. The amount $A(t)$ of radioactive material at time t is modeled by

$$A'(t) = kA, A(0) = e^3$$

together with the given half-life of the material,

$$A(\ln(2)) = \frac{1}{2}e^3.$$

Solve this (as in the text) to obtain

$$A(t) = \left(\frac{1}{2}\right)^{t/\ln(2)} e^3.$$

Then

$$A(3) = e^3 \left(\frac{1}{2}\right)^{3/\ln(2)} = 1 \text{ tonne}.$$

19. The problem is like Problem 18, and we find that the amount of Uranium-235 at time t is

$$U(t) = 10 \left(\frac{1}{2}\right)^{t/(4.5(10^9))},$$

with t in years. Then

$$U(10^9) = 10 \left(\frac{1}{2}\right)^{1/4.5} \approx 8.57 \text{ kg}.$$

20. At time t there will be $A(t) = 12e^{kt}$ grams, and $A(4) = 12e^{4k} = 9.1$. Solve this for k to get

$$k = \frac{1}{4}\ln\left(\frac{9.1}{12}\right).$$

The half-life of this element is the time t^* it will take for there to be 6 grams, so

$$A(t^*) = 6 = 12e^{\ln(9.1/12)t^*/4}.$$

Solve this to get

$$t^* = \frac{4\ln(1/2)}{\ln(9.1/12)} \approx 10.02 \text{ minutes}.$$

21. Let

$$I(x) = \int_0^\infty e^{-t^2-(x/t)^2}\, dt.$$

The integral we want is $I(3)$. Compute

$$I'(x) = -2x \int_0^\infty \frac{1}{t^2} e^{-t^2-(x/t)^2}\, dt.$$

Let $u = x/t$, so $t = x/u$ and

$$dt = -\frac{x}{u^2}\,du.$$

Then

$$\begin{aligned} I'(x) &= -2x \int_\infty^0 \left(\frac{u^2}{x^2}\right) e^{-(x/u)^2 - u^2} \frac{-x}{u^2}\,du \\ &= -2I(x). \end{aligned}$$

Then $I(x)$ satisfies the separable differential equation $I' = -2I$, with general solution of the form $I(x) = ce^{-2x}$. Now observe that

$$I(0) = \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} = c,$$

in which we used a standard integral that arises often in statistics. Then

$$I(x) = \frac{\sqrt{\pi}}{2} e^{-2x}.$$

Finally, put $x = 3$ for the particular integral of interest:

$$I(3) = \int_0^\infty e^{-t^2 - (9/t)^2}\,dt = \frac{\sqrt{\pi}}{2} e^{-6}.$$

22. Begin with the logistic equation

$$P'(t) = aP(t) - bP(t)^2,$$

in which a and b are positive constants. Then

$$\frac{dP}{dt} = P(a - bP)$$

so

$$\frac{1}{P(a - bP)}\,dP = dt$$

and the variables are separated. To make the integration easier, write this equation as

$$\left(\frac{1}{a}\frac{1}{P} + \frac{b}{a}\frac{1}{a - bP}\right) dP = dt.$$

Integrate to obtain

$$\frac{1}{a}\ln(P) - \frac{b}{a}\ln(a - bP) = t + c.$$

if $P(t) > -$ and $a - bP(t) > 0$. Using properties of the logarithm, we can write this equation as

$$\ln\left(\frac{P}{a - bP}\right) = at + k,$$

in which $k = ac$ is still constant. Then

$$\frac{P}{a - bP} = e^{at+k} = e^k e^{at} = Ke^{at},$$

in which $K = e^k$ is a positive constant. Now suppose the initial population (say at time zero) is p_0. Then $P(0) = p_0$ and

$$\frac{p_0}{a - bp_0} = K.$$

We now have

$$\frac{P}{a - bP} = \frac{p_0}{a - bp_0} e^{at}.$$

It is a straightforward algebraic manipulation to solve this for $P(t)$:

$$P(t) = \frac{ap_0}{a - bp_0 + bp_0} e^{at}.$$

This is the solution of the logistic equation with $P(0) = p_0$.

Because $a - bp_0 > 0$ by assumption, then

$$bp_0 e^{at} < a - bp_0 + bpe^{at},$$

so

$$P(t) < \frac{ap_0}{bp_0 e^{at}} e^{at} = \frac{a}{b}.$$

This means that this population function is bounded above. Further, by multiplying the numerator and denominator of $P(t)$ by $e(-at$, we have

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{ap_0}{(a - bp_0)e^{-at} + bp_0}$$
$$= \lim_{t\to\infty} \frac{ap_0}{bp_0} = \frac{a}{b}.$$

23. With a and b as given, and $p_0 = 3,929,214$ (the population in 1790), the logistic population function for the United States is

$$P(t) = \frac{123,141.5668}{0.03071576577 + 0.0006242342283e^{0.03134t}} e^{0.03134t}.$$

If we attempt an exponential model $Q(t) = Ae^{kt}$, then take $A = Q(0) = 3,929,214$, the population in 1790. To find k, use the fact that

$$Q(10) = 5308483 = 3929214e^{10k}$$

and we can solve for k to get

$$k = \frac{1}{10} \ln\left(\frac{5308483}{3929214}\right) \approx 0.03008667012.$$

year	population	$P(t)$	percent error	$Q(t)$	percent error
1790	3,929,213	3,929,214	0	3,929,214	0
1800	5,308,483	5,336,313	0.52	5,308,483	0
1810	7,239,881	7,228,171	-0.16	7,179,158	-0.94
1820	9,638,453	9,757,448	1.23	7,179,158	0.53
1830	12,886,020	13,110,174	1.90	13,000,754	1.75
1840	17,169,453	17,507,365	2.57	17,685,992	3.61
1850	23,191,876	23.193,639	0.008	23,894,292	3.03
1860	31,443,321	30,414,301	-3.27	32,281,888	2.67
1870	38,558,371	39,374,437	2.12	43,613,774	13.11
1880	50,189,209	50,180,383	-0.018	58,923,484	17.40
1890	62,979766	62,772,907	-0.33	79,073,491	26.40
1900	76,212,168	76,873,907	0.87	107,551,857	41.12
1910	92,228,496	91,976,297	-0.27	145,303,703	57.55
1920	106,021,537	107,398,941	1.30	196,312,254	83.16
1930	123,202,624	122,401,360	-0.65		
1940	132,164,569	136,329,577	3.15		
1950	151,325,798	148,679,224	-1.75		
1960	179,323,175	150,231,097	-11.2		
1970	203,302,031	167,943,428	-17.39		
1980	226,547,042	174,940,040	-22.78		

Table 1.1: Census data for Problem 23

The exponential model, using these two data points (1790 and 1800 populations), is

$$Q(t) = 3929214e^{0.03008667012t}.$$

Table 1.1 uses $Q(t)$ and $P(t)$ to predict later populations from these two initial figures. The logistic model remains quite accurate until about 1960, at which time it loses accuracy quickly. The exponential model becomes quite inaccurate by 1870, after which the error becomes so large that it is not worth computing further. Exponential models do not work well over time with complex populations, such as fish in the ocean or countries throughout the world.

1.2 The Linear First-Order Equation

1. With $p(x) = -3/x$, and integrating factor is

$$e^{\int(-3/x)\,dx} = e^{-3\ln(x)} = x^{-3}$$

for $x > 0$. Multiply the differential equation by x^{-3} to get

$$x^{-3}y' - 3x^{-4} = 2x^{-1}.$$

or

$$\frac{d}{dx}(x^{-3}y) = \frac{2}{x}.$$

Integrate to get

$$x^{-3}y = 2\ln(x) + c,$$

with c an arbitrary constant. For $x > 0$ we have a general solution

$$y = 2x^3\ln(x) + cx^3.$$

In the last integration, we can allow $x < 0$ by replacing $\ln(x)$ with $\ln|x|$ to derive the solution

$$y = 2x^3\ln|x| + cx^3$$

for $x \neq 0$.

2. $e^{\int dx} = e^x$ is an integrating factor. Multiply the differential equation by e^x to get

$$y'e^x + ye^x = \frac{1}{2}(e^{2x} - 1).$$

Then

$$(e^x y)' = \frac{1}{2}(e^{2x} - 1)$$

and an integration gives us

$$e^x y = \frac{1}{4}e^{2x} - \frac{1}{2}x + c.$$

Then

$$y = \frac{1}{4}e^x - \frac{1}{2}xe^{-x} + ce^{-x}$$

is a general solution, with c an arbitrary constant.

3. $e^{\int 2\,dx} = e^{2x}$ is an integrating factor. Multiply the differential equation by e^{2x}:

$$y'e^{2x} + 2ye^{2x} = xe^{2x},$$

or

$$(e^{2x}y)' = xe^{2x}.$$

Integrate to get

$$e^{2x}y = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c.$$

giving us the general solution

$$y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}.$$

4. For an integrating factor, compute

$$e^{\int \sec(x)\,dx} = e^{\ln|\sec(x)+\tan(x)|} = \sec(x) + \tan(x).$$

Multiply the differential equation by this integrating factor:

$$\begin{aligned} &y'(\sec(x)+\tan(x)) + \sec(x)(\sec(x)+\tan(x))y \\ &= y'(\sec(x)+\tan(x)) + (\sec(x)\tan(x)+\sec^2(x))y \\ &= ((\sec(x)+\tan(x))y)' \\ &= \cos(x)(\sec(x)+\tan(x)) \\ &= 1+\sin(x). \end{aligned}$$

We therefore have

$$((\sec(x)+\tan(x))y)' = 1+\sin(x).$$

Integrate to get

$$y(\sec(x)+\tan(x)) = x - \cos(x) + c.$$

Then

$$y = \frac{x-\cos(x)+c}{\sec(x)+\tan(x)}.$$

This is a general solution. If we wish, we can also observe that

$$\frac{1}{\sec(x)+\tan(x)} = \frac{\cos(x)}{1+\sin(x)}$$

to obtain

$$\begin{aligned} y &= (x-\cos(x)+c)\left(\frac{\cos(x)}{1+\sin(x)}\right) \\ &= \frac{x\cos(x)-\cos^2(x)+c\cos(x)}{1+\sin(x)}. \end{aligned}$$

5. First determine the integrating factor

$$e^{\int -2\,dx} = e^{-2x}.$$

Multiply the differential equation by e^{-2x} to get

$$(e^{-2x}y)' = -8x^2e^{-2x}.$$

Integrate to get

$$e^{-2x}y = \int -8x^2e^{-2x}\,dx = 4x^2e^{-2x} + 4xe^{-2x} + 2e^{-2x} + c.$$

This yields the general solution

$$y = 4x^2 + 4x + 2 + ce^{2x}.$$

6. $e^{\int 3\,dx} = e^{3x}$ is an integrating factor. Multiply the differential equation by e^{3x} to get
$$(e^{3x}y)' = 5e^{5x} - 6e^{3x}.$$
Integrate this equation:
$$e^{3x}y = e^{5x} - 2e^{3x} + c.$$
Now we have a general solution
$$y = e^{2x} - 2 + ce^{-3x}.$$
We need
$$y(0) = 2 = 1 - 2 + c,$$
so $c = 3$. The unique solution of the initial value problem is
$$y = e^{2x} + 3e^{-3x} - 2.$$

7. $x - 2$ is an integrating factor for the differential equation because
$$e^{\int (1/(x-2))\,dx} = e^{\ln(x-2)} = x - 2.$$
Multiply the differential equation by $x - 2$ to get
$$((x-2)y)' = 3x(x-2).$$
Integrate to get
$$(x-2)y = x^3 - 3x^2 + c.$$
This gives us the general solution
$$y = \frac{1}{x-2}(x^3 - 3x^2 + c).$$
Now we need
$$y(3) = 27 - 27 + c = 4,$$
so $c = 4$ and the solution of the initial value problem is
$$y = \frac{1}{x-2}(x^3 - 3x^2 + 4).$$

8. $e^{\int (-1)\,dx} = e^{-x}$ is an integrating factor. Multiply the differential equation by e^{-x} to get:
$$(ye^{-x})' = 2e^{3x}.$$
Integrate to get
$$ye^{-x} = \frac{2}{3}e^{3x} + c,$$

and we have the general solution

$$y = \frac{2}{3}e^{4x} + ce^{x}.$$

We need

$$y(0) = \frac{2}{3} + c = -3,$$

so $c = -11/3$ and the initial value problem has the solution

$$y = \frac{2}{3}e^{4x} - \frac{11}{3}e^{x}.$$

9. First derive the integrating factor

$$e^{\int (2/(x+1))\,dx} = e^{2\ln(x+1)} = e^{\ln((x+1)^2)} = (x+1)^2.$$

Multiply the differential equation by $(x+1)^2$ to obtain

$$\left((x+1)^2 y\right)' = 3(x+1)^2.$$

Integrate to obtain

$$(x+1)^2 y = (x+1)^3 + c.$$

Then

$$y = x + 1 + \frac{c}{(x+1)^2}.$$

Now

$$y(0) = 1 + c = 5$$

so $c = 4$ and the initial value problem has the solution

$$y = x + 1 + \frac{4}{(x+1)^2}.$$

10. An integrating factor is

$$e^{\int (5/9x)\,dx} = e^{(5/9)\ln(x)} = e^{\ln(x^{5/9})} = x^{5/9}.$$

Multiply the differential equation by $x^{5/9}$ to get

$$(yx^{5/9})' = 3x^{32/9} + x^{14/9}.$$

Integrate to get

$$yx^{5/9} = \frac{27}{41}x^{41/9} + \frac{9}{23}x^{23/9} + c.$$

Then

$$y = \frac{27}{41}x^4 + \frac{9}{23}x^2 + cx^{-5/9}.$$

Finally, we need

$$y(-1) = \frac{27}{41} + \frac{9}{23} - c = 4.$$

Then $c = -2782/943$, so the initial value problem has the solution

$$y = \frac{23}{41}x^4 + \frac{9}{23}x^2 - \frac{2782}{943}x^{-5/9}.$$

11. Let (x, y) be a point on the curve. The tangent line at (x, y) must pass through $(0, 2x^2)$, and so has slope

$$y' = \frac{y - 2x^2}{x}.$$

This is the linear differential equation

$$y' - \frac{1}{x}y = -2x.$$

An integrating factor is

$$e^{-\int (1/x)\,dx} = e^{-\ln(x)} = e^{\ln(1/x)} = \frac{1}{x},$$

so multiply the differential equation by $1/x$ to get

$$\frac{1}{x}y' - \frac{1}{x^2}y = -2.$$

This is

$$\left(\frac{1}{x}y\right)' = -2.$$

Integrate to get

$$\frac{1}{x}y = -2x + c.$$

Then

$$y = -2x^2 + cx,$$

in which c can be any number.

12. Let $A(t)$ be the number of pounds of salt in the tank at time $t \geq 0$. Then

$$\begin{aligned}\frac{dA}{dt} &= \text{ rate salt is added } - \text{ rate salt is removed} \\ &= 6 - 2\left(\frac{A(t)}{50+t}\right).\end{aligned}$$

We must solve this subject to the initial condition $A(0) = 25$. The differential equation is

$$A' + \frac{2}{50+t}A = 6,$$

which is linear with integrating factor

$$e^{\int 2/(50+t)\,dt} = e^{2\ln(50+t)} = (50+t)^2.$$

Multiply the differential equation by $(50+t)^2$ to get

$$(50+t)^2 A' + 2(50+t)A = 6(50+t)^2.$$

This is

$$\left((50+t)^2A\right)' = 6(50+t)^2.$$

Integrate this equation to get

$$(50+t)^2A = 2(50+t)^3 + c,$$

which we will write as

$$A(t) = 2(50+t) + \frac{c}{(50+t)^2}.$$

We need c so that

$$A(0) = 100 + \frac{c}{2500} = 25,$$

so $c = 187,500$. The number of pounds of salt in the tank at time t is

$$A(t) = 2(50+t) - \frac{187,500}{(50+t)^2}.$$

13. Let $A_1(t)$ and $A_2(t)$ be the number of pounds of salt in tanks 1 and 2, respectively, at time t. Then

$$A_1'(t) = \frac{5}{2} - \frac{5A_1(t)}{100}; A_1(0) = 20$$

and

$$A_2'(t) = \frac{5A_1(t)}{100} - \frac{5A_2(t)}{150}; A_2(0) = 90.$$

Solve the linear initial value problem for $A_1(t)$ to get

$$A_1(t) = 50 - 30e^{-t/20}.$$

Substitute this into the differential equation for $A_2(t)$ to get

$$A_2' + \frac{1}{30}A_2 = \frac{5}{2} - \frac{3}{2}e^{-t/20}; A_2(0) = 90.$$

Solve this linear problem to obtain

$$A_2(t) = 75 + 90e^{-t/20} - 75e^{-t/30}.$$

Tank 2 has its minimum when $A_2'(t) = 0$, and this occurs when

$$2.5e^{-t/30} - 4.5e^{-t/20} = 0.$$

This occurs when $e^{t/60} = 9/5$, or $t = 60\ln(9/5)$. Then

$$A_2(t)_{\text{min}} = A_2(60\ln(9/5)) = \frac{5450}{81}$$

pounds.

1.3 Exact Equations

In these problems it is assumed that the differential equation has the form $M(x,y) + N(x,y)y' = 0$, or, in differential form, $M(x,y)\,dx + N(x,y)\,dy = 0$.

1. With $M(x,y) = 2y^2 + ye^{xy}$ and $N(x,y) = 4xy + xe^{xy} + 2y$. Then

$$\frac{\partial N}{\partial x} = 4y + e^{xy} + xye^{xy} = \frac{\partial M}{\partial y}$$

for all (x,y), so the differential equation is exact on the entire plane. A potential function $\varphi(x,y)$ must satisfy

$$\frac{\partial \varphi}{\partial x} = M(x,y) = 2y^2 + ye^{xy}$$

and

$$\frac{\partial \varphi}{\partial y} = N(x,y) = 4xy + xe^{xy} + 2y.$$

Choose one to integrate. If we begin with $\partial\varphi/\partial x = M$, then integrate with respect to x to get

$$\varphi(x,y) = 2xy^2 + e^{xy} + \alpha(y),$$

with $\alpha(y)$ the "constant" of integration with respect to x. Then we must have

$$\frac{\partial \varphi}{\partial y} = 4xy + xe^{xy} + \alpha'(y) = 4xy + xe^{xy} + 2y.$$

This requires that $\alpha'(y) = 2y$, so we can choose $\alpha(y) = y^2$ to obtain the potential function

$$\varphi(x,y) = 2xy^2 + e^{xy} + y^2.$$

The general solution is defined implicitly by the equation

$$2xy^2 + e^{xy} + y^2 = c,,$$

with c an arbitrary constant.

2. $\partial M/\partial y = 4x = \partial N/\partial x$ for all (x,y), so this equation is exact on the entire plane. For a potential function, we can begin by integrating

$$\frac{\partial \varphi}{\partial y} = 2x^2 + 3y^2$$

to get

$$\varphi(x,y) = 2x^2y + y^3 + c(x).$$

Then

$$\frac{\partial \varphi}{\partial x} = 4xy + 2x = 4xy + c'(x).$$

Then $c'(x) = 2x$ so we can choose $c(x) = x^2$ to obtain the potential function

$$\varphi(x, y) = 2x^2y + y^3 + x^2.$$

The general solution is defined implicitly by

$$2x^2y + y^3 + x^2 = k,$$

with k an arbitrary constant.

3. $\partial M/\partial y = 4x + 2x^2$ and $\partial N/\partial x = 4x$, so this equation is not exact (on any rectangle).

4.
$$\frac{\partial M}{\partial y} = -2\sin(x + y) + 2x\cos(x + y) = \frac{\partial N}{\partial x},$$

for all (x, y), so this equation is exact on the entire plane. Integrate $\partial\varphi/\partial x = M$ or $\partial\varphi/\partial y = N$ to obtain the potential function

$$\varphi(x, y) = 2x\cos(x + y).$$

The general solution is defined implicitly by

$$2x\cos(x + y) = k$$

with k an arbitrary constant.

5. $\partial M/\partial y = 1 = \partial N/\partial x$, for $x \neq 0$, so this equation is exact on the plane except at points $(0, y)$. Integrate $\partial\varphi/\partial x = M$ or $\partial\varphi/\partial y = N$ to find the potential function

$$\varphi(x, y) = \ln|x| + xy + y^3$$

for $x \neq 0$. The general solution is defined by an equation

$$\ln|x| + xy + y^3 = k.$$

6. For the equation to be exact, we need

$$\frac{\partial M}{\partial y} = \alpha x y^{\alpha-1} = \frac{\partial N}{\partial x} = -2xy^{\alpha-1}.$$

This will hold if $\alpha = -2$. With this choice of α, the (exact) equation is

$$3x^2 + xy^{-2} - x^2y^{-3}y' = 0.$$

Routine integrations produce a potential function

$$\varphi(x, y) = x^3 + \frac{x^2}{2y^2}.$$

The general solution is defined by the equation

$$x^3 + \frac{x^2}{2y^2} = k,$$

for $y \neq 0$.

7. For this equation to be exact, we need

$$\frac{\partial M}{\partial y} = 6xy^2 - 3 = \frac{\partial N}{\partial x} = -3 - 2\alpha xy^2.$$

This will be true if $\alpha = -3$. By integrating, we find a potential function

$$\varphi(x, y) = x^2y^3 - 3xy - 3y^2$$

and a general solution is defined implicitly by

$$x^2y^3 - 3xy - 3y^2 = k.$$

8. We have

$$\frac{\partial M}{\partial y} = 2 - 2y\sec^2(xy^2) - 2xy^3\sec^2(xy^2)\tan(xy^2) = \frac{\partial N}{\partial x},$$

for all (x, y), so this equation is exact over the entire plane. By integrating $\partial\varphi/\partial x = 2y - y^2\sec^2(xy^2)$ with respect to x, we find that

$$\varphi(x, y) = 2xy - \tan(xy^2) + c(y).$$

Then

$$\begin{aligned}\frac{\partial\varphi}{\partial y} &= 2x - 2xy\sec^2(xy^2) \\ &= 2x - 2xy\sec^2(xy^2) + c'(y).\end{aligned}$$

Then $c'(y) = 0$ and we can choose $c(y) = 0$ to obtain the potential function

$$\varphi(x, y) = 2xy - \tan(xy^2).$$

A general solution is defined implicitly by

$$2xy - \tan(xy^2) = k.$$

For the solution satisfying $y(1) = 2$, put $x = 1$ and $y = 2$ into this implicitly defined solution to get

$$4 - \tan(4) = k.$$

The solution of the initial value problem is defined implicitly by

$$2xy - \tan(xy^2) = 4 - \tan(4).$$

9. Because $\partial M/\partial y = 12y^2 = \partial N/\partial x$, this equation is exact for all (x, y). Straightforward integrations yield the potential function

$$\varphi(x, y) = 3xy^4 - x.$$

A general solution is defined implicitly by

$$3xy^4 - x = k.$$

To satisfy the condition $y(1) = 2$, we must choose k so that

$$48 - 1 = k,$$

so $k = 47$ and the solution of the initial value problem is specified by the equation

$$3xy^4 - x = 47.$$

In this case we can actually write this solution explicitly with y in terms of x.

10. First,

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{1}{x}e^{y/x} - \frac{1}{x}e^{y/x} - \frac{y}{x^2}e^{y/x} \\ &= -\frac{y}{x^2}e^{y/x} = \frac{\partial N}{\partial x},\end{aligned}$$

so the equation is exact for all (x, y) with $x \neq 0$. For a potential function, we can begin with

$$\frac{\partial \varphi}{\partial y} = e^{y/x}$$

and integrate with respect to y to get

$$\varphi(x, y) = xe^{y/x} + c(x).$$

Then we need

$$\frac{\partial \varphi}{\partial x} = 1 + e^{y/x} - \frac{y}{x}e^{y/x} = e^{y/x} - \frac{y}{x}e^{y/x} + c'(x).$$

This requires that $c'(x) = 1$ and we can choose $c(x) = x$. Then

$$\varphi(x, y) = xe^{y/x} + x.$$

The general solution of the differential equation is implicitly defined by

$$xe^{y/x} + x = k.$$

To have $y(1) = -5$, we must choose k so that

$$e^{-5} + 1 = k.$$

The solution of the initial value problem is given by

$$xe^{y/x} + x = 1 + e^{-5}.$$

This can be solve for y to obtain the explicit solution

$$y = x \ln\left(\frac{1 + e^{-5}}{x + 1}\right)$$

for $x + 1 > 0$.

11. First,

$$\frac{\partial M}{\partial y} = -2x\sin(2y-x) - 2\cos(2y-x) = \frac{\partial N}{\partial x},$$

so the differential equation is exact for all (x, y). For a potential function, integrate

$$\frac{\partial \varphi}{\partial y} = -2x\cos(2y-x)$$

with respect to y to get

$$\varphi(x,y) = -x\sin(2y-x) + c(x).$$

Then we must have

$$\begin{aligned}\frac{\partial \varphi}{\partial x} &= x\cos(2y-x) - \sin(2y-x)\\ &= x\cos(2y-x) - \sin(2y-x) + c'(x).\end{aligned}$$

Then $c'(x) = 0$ and we can take $c(x)$ to be any constant. Choosing $c(x) = 0$ yields

$$\varphi(x,y) = -x\sin(2y-x).$$

The general solution is defined implicitly by

$$-x\sin(2y-x) = k.$$

To satisfy $y(\pi/12) = \pi/8$, we need

$$-\frac{\pi}{12}\sin(\pi/6) = k,$$

so choose $k = -\pi/24$ to obtain the solution defined by

$$-x\sin(2y-x) = -\frac{\pi}{24}$$

which of course is the same as

$$x\sin(2y-x) = \frac{\pi}{24}.$$

We can also write

$$y = \frac{1}{2}\left(x + \arcsin\left(\frac{\pi}{24x}\right)\right)$$

for $x \neq 0$.

12.

$$\frac{\partial M}{\partial y} = e^y = \frac{\partial N}{\partial x}$$

so the differential equation is exact. Integrate

$$\frac{\partial \varphi}{\partial x} = e^y$$

with respect to x to get

$$\varphi(x, y) = xe^y + c(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = xe^y + c'(y) = xe^y - 1,$$

so $c'(y) = -1$ and we can let $c(y) = -y$. This gives us the potential function

$$\varphi(x, y) = xe^y - y.$$

The general solution is given by

$$xe^y - y = k.$$

For $y(5) = 0$ we need

$$5 - 0 = k$$

so $k = 5$ and the solution of the initial value problem is given by

$$xe^y - y = 5.$$

13. $\varphi + c$ is also a potential function if φ is because

$$\frac{\partial \varphi}{\partial x} = \frac{\partial(\varphi + c)}{\partial x}$$

and

$$\frac{\partial \varphi}{\partial y} = \frac{\partial(\varphi + c)}{\partial y}.$$

The function defined implicitly by

$$\varphi(x, y) = k$$

is the same as that defined by

$$\varphi(x, y) + c = k$$

if k is arbitrary.

14. (a)

$$\frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = -1$$

so this equation is not exact over any rectangle in the plane.

(b) Multiply the differential equation by x^{-2} to obtain

$$yx^{-2} - x^{-1}y' = 0.$$

This is exact because

$$\frac{\partial M^*}{\partial y} = x^{-2} = \frac{\partial N^*}{\partial x}.$$

This new equation has potential function $\varphi(x, y) = -yx^{-1}$ and so has general solution defined implicitly by

$$-\frac{y}{x} = k.$$

This also defines a general solution of the original differential equation.

(c) Multiply the differential equation by y^{-2} to obtain

$$y^{-1} - xy^{-2}y' = 0.$$

This is exact on any region of the plane not containing $y = 0$, because

$$\frac{\partial M^{**}}{\partial y} = -y^{-2} = \frac{\partial N^{**}}{\partial x}.$$

The new equation has potential function $\varphi(x, y) = xy^{-1}$, so its general solution is defined implicitly by

$$xy^{-1} = k.$$

It is easy to check that this also defines a solution of the original differential equation.

(d) Multiply the differential equation by xy^{-2} to obtain

$$xy^{-2} - x^2y^{-3}y' = 0.$$

This is exact (on any region not containing $y = 0$) because

$$\frac{\partial M^{***}}{\partial y} = -2xy^{-3} = \frac{\partial N^{***}}{\partial x}.$$

Integrate $\partial\varphi/\partial x = xy^{-2}$ with respect to x to obtain

$$\varphi(x, y) = \frac{1}{2}x^2y^{-2} + c(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = -x^{-2}y^{-3} + c'(y) = -x^2y^{-3},$$

so $c' = 0$ and we can choose $c(y) = 0$. Then

$$\varphi(x, y) = \frac{1}{2}x^2y^{-2}$$

and we can define a general solution of this differential equation as

$$x^2y^{-2} = k.$$

Here we absorbed the factor of 1/2 into the arbitrary constant c. This again defines a solution of the original differential equation.

(e) The original differential equation can be written as the linear equation

$$y' - \frac{1}{x}y = 0.$$

This has integrating factor

$$e^{\int -(1/x)\,dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1}.$$

Multiply the differential equation by x^{-1} to write this equation as

$$(x^{-1}y)' = 0,$$

so $x^{-1}y = c$ implicitly defines the general solution.

(f) The methods of (b) through (e) yield the same general solution. For example, in (b) we obtained $-yx^{-1} = c$, which we can write as $y = -cx$. Because c is an arbitrary constant, this general solution can be written $y = kx$. And in (d) we obtained $x^2y^{-2} = c$, and this gives the same solutions as $y^2 = cx^2$, or $y = kx$.

15. First,

$$\frac{\partial M}{\partial y} = x - \frac{3}{2}y^{-5/2} \text{ and } \frac{\partial N}{\partial x} = 2x.$$

and these are not equal on any rectangle in the plane.

In differential form, the differential equation is

$$(xy + y^{-3/2})\,dx + x^2\,dy = 0.$$

Multiply this equation by x^ay^b to get

$$(x^{a+1}y^{b+1} + x^ay^{b-3/2})\,dx + x^{a+2}y^b\,dy = 0 = M^*\,dx + N^*\,dy.$$

For this to be exact, we need

$$\begin{aligned}\frac{\partial M^*}{\partial y} &= (b+1)x^{a+1}y^b + \left((b - \frac{3}{2}\right)x^ay^{b-5/2} \\ &= \frac{\partial N^*}{\partial x} = (a+2)x^{a+1}y^b.\end{aligned}$$

Divide this equation by x^ay^b to get

$$(b+1)x + \left(b - \frac{3}{2}\right)y^{-5/2} = (a+2)x.$$

This will hold for all x and y if we let $b = 3/2$ and then choose a and b so that $b + 1 = a + 2$. Thus choose

$$a = \frac{1}{2} \text{ and } b = \frac{3}{2}$$

to get the integrating factor $\mu(x,y) = x^{1/2}b^{3/2}$. Multiply the original differential equation by this to get

$$(x^{3/2}y^{5/2} + x^{1/2})\,dx + x^{5/2}y^{1/2}\,dy = 0.$$

To find a potential function, integrate

$$\frac{\partial \varphi}{\partial y} = x^{5/2}y^{3/2}$$

with respect to y to get

$$\varphi(x,y) = \frac{2}{5}x^{5/2}y^{5/2} + c(x).$$

Then we need

$$\frac{\partial \varphi}{\partial x} = x^{3/2}y^{5/2} + c'(x) = x^{3/2}y^{5/2} + x^{1/2}.$$

Therefore $c'(x) = x^{1/2}$, so $c(x) = 2x^{3/2}/3$ and

$$\varphi(x) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2}.$$

The general solution of the original differential equation is given implicitly by

$$\frac{2}{5}(xy)^{5/2} + \frac{2}{3}x^{3/2} = k.$$

In this we must have $x \neq 0$ and $y \neq 0$ to ensure that the integrating factor $\mu(x,y) \neq 0$.

16. It is routine to verify that the differential equation is not exact. To find an integrating factor, first multiply by $x^a y^b$ to get

$$(2x^a y^{b+2} - 9x^{a+1}y^{b+1})\,dx + (3x^{a+1}y^{b+1} - 6x^{a+2}y^b)\,dy = 0.$$

For this to be exact, we must have

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2(b+2)x^a y^{b+1} - 9(b+1)x^{a+1}y^b \\ &= \frac{\partial N}{\partial x} = 3(a+1)x^a y^{b+1} - 6(a+2)x^{+1}y^b.\end{aligned}$$

Divide by $x^a y^b$ and rearrange terms to obtain

$$(2(b+2) - 3(a+1))y = (9(b+1) - 6(a+2))x.$$

Because x and y are independent, both coefficients must be zero:

$$2(b+2) - 3(a+1) = 0 \text{ and } 9(b+1) - 6(a+2) = 0.$$

Solve these to get $a = b = 1$, so $\mu(x, y) = xy$ is an integrating factor. Multiply the differential equation by xy to obtain, in differential form,

$$(2xy^3 - 9x^2y^2)\,dx + (3x^2y^2 - 6x^3y)\,dy = 0.$$

This equation is exact. For a potential function, integrate

$$\frac{\partial \varphi}{\partial x} = 2xy^3 - 9x^2y^2$$

with respect to x to get

$$\varphi(x, y) = x^2y^3 - 3x^3y^2 + c(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = 3x^2y^2 - 6x^3y + c'(y) = 3x^2y^2 - 6x^3y.$$

Then $c'(y) = 0$ and we can choose $c(y) = 0$ got s potential function

$$\varphi(x, y) = x^2y^3 - 3x^3y^2.$$

A general solution of this equation, and also the original equation, is given by

$$x^2y^3 - 3x^3y^2 = k.$$

This requires that $\mu(x, y) \neq 0$.

1.4 Homogeneous, Bernoulli and Riccati Equations

1. This is a Riccati equation and one solution (by inspection) is $S(x) = x$. Let $y = x + 1/z$ to obtain

$$2 - \frac{1}{z^2}z' = \frac{1}{x^2}\left(x + \frac{1}{z}\right)^2 - \frac{1}{x}\left(x + \frac{1}{z}\right) + 1.$$

This simplifies to

$$z' + \frac{1}{x}z = -\frac{1}{x^2},$$

a linear equation with integrating factor

$$e^{\int (1/x)\,dx} = e^{\ln(x)} = x.$$

The differential equation for z can therefore be written

$$(xz)' = -\frac{1}{x}.$$

Integrate to get

$$xz = -\ln(x) + c,$$

so

$$z = -\frac{\ln(x)}{x} + \frac{c}{x} = \frac{c - \ln(x)}{x}.$$

for $x > 0$. Then

$$y = x + \frac{1}{z} = x + \frac{x}{c - \ln(x)}$$

for $x > 0$.

2. This is a Bernoulli equation with $\alpha = -4/3$. Put $v = y^{7/3}$, so $y = v^{7/3}$. Substitute this into the differential equation to get

$$\frac{3}{7}v^{-4/7}v' + \frac{7}{3x}v^{3/7} = \frac{14}{3x^2}.$$

This simplifies to the linear differential equation

$$v' + \frac{7}{3x}v = \frac{14}{3x^2}$$

which has integrating factor

$$e^{\int 7/3x\,dx} = e^{(7/3)\ln(x)} = e^{\ln(x^{7/3})} = x^{7/3}$$

for $x > 0$. Multiply the differential equation by $x^{7/3}$ to get

$$(x^{7/3}v)' = \frac{14}{3}x^{1/3}.$$

Integrate to get

$$vx^{7/3} = \frac{7}{2}x^{4/3} + c.$$

Because $v = y^{7/3}$, this gives us

$$2y^{7/3}x^{7/3} - 7x^{4/3} = k,$$

in which $k = 2c$ is an arbitrary constant. This equation implicitly defines the general solution.

3. This is a Bernoulli equation with $\alpha = 2$, so let $v = y^{1-\alpha} = y^{-1}$ for $y \neq 0$ and $y = 1/v$. Compute

$$y' = \frac{dy}{dv}\frac{dv}{dx} = -\frac{1}{v^2}xv'.$$

The differential equation becomes

$$-\frac{1}{v^2}v' + \frac{x}{v} = \frac{x}{v^2}.$$

This is

$$v' - xv = -x,$$

a linear equation with integrating factor $e^{-x^2/2}$. We can therefore write

$$(e^{-x^2/2}v)' = -xe^{-x^2/2}.$$

Integrate to get

$$e^{-x^2/2}v = e^{-x^2/2} + c,$$

so

$$v = 1 + ce^{-x^2/2}.$$

The original differential equation has the general solution

$$y = \frac{1}{v} = \frac{1}{1 + ce^{-x^2/2}},$$

in which c is an arbitrary constant.

4. This equation is homogeneous. With $y = ux$ we obtain

$$u + xu' = u + \frac{1}{u}.$$

Then

$$x\frac{du}{dx} = \frac{1}{u},$$

a separable equation. In differential form, this is

$$u\,du = \frac{1}{x}\,dx.$$

Integrate to get

$$\frac{1}{2}u^2 = \ln|x| + c.$$

Then

$$\frac{1}{y^2}x^2 = 2\ln|x| + k,$$

where $k = 2c$ is an arbitrary constant. This implicitly defines the general solution.

5. This differential equation is homogeneous and setting $y = ux$ gives us

$$u + xu' = \frac{u}{1+u}.$$

This is the separable equation

$$x\frac{du}{dx} = \frac{u}{1+u} - u$$

which, in terms of x and y, is

$$\left(\frac{1}{u^2} + \frac{1}{u}\right)du = -\frac{1}{x}\,dx.$$

Integrate to get

$$\frac{1}{u} + \ln|u| = -\ln|x| + c.$$

With $u = y/x$ this reduces to

$$-x + y\ln|y| = cy,$$

with c an arbitrary constant.

6. This is a Riccati equation and one solution (by inspection) is $S(x) = 4$. After some routine computation we obtain the general solution

$$y = 4 + \frac{6x^3}{c - x^3}.$$

7. The differential equation is exact, with general solution defined implicitly by

$$xy - x^2 - y^2 = c.$$

8. The differential equation is homogeneous, and $y = ux$ yields the general solution defined by

$$\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = cx.$$

9. The differential equation is of Bernoulli type with $\alpha = -3/4$. The general solution is defined by

$$5(xy)^{7/4} + 7x^{-5/4} = c.$$

10. The differential equation is homogeneous and $y = ux$ leads to the separable differential equation

$$\frac{1 - u + u^2}{}du = \frac{1}{x}\,dx.$$

Integrate and set $u = y/x$ to obtain the general solution implicitly defined by

$$\frac{2}{\sqrt{3}}\arctan\left(\frac{2y - x}{\sqrt{3}x}\right) = \ln|x| + c.$$

11. The equation is Bernoulli with $\alpha = 2$ and the change of variables $v = y^{-1}$ leads to the general solution

$$y = 2 + \frac{2}{cx^2 - 1}.$$

12. The equation is homogeneous and $y = ux$ leads to the general solution defined by

$$\frac{1}{2}\frac{x^2}{y^2} = \ln|x| + c.$$

13. The differential equation is Riccati and one solution is $S(x) = e^x$. A general solution is given explicitly by

$$y = \frac{2e^x}{ce^{2x} - 1}.$$

14. The equation is Bernoulli with $\alpha = 2$ and a general solution is given by

$$y = \frac{2}{3 + cx^2}.$$

15. For the first part,

$$F\left(\frac{ax + by + c}{dx + py + r}\right) = F\left(\frac{a + b(y/x)c/x}{d + p(y/x) + r/x}\right) = f\left(\frac{y}{x}\right)$$

if and only if $c = r = 0$.

Next, suppose $x = X + h$ and $y = Y + k$. Then

$$\begin{aligned} \frac{dY}{dX} &= F\left(\frac{a(X + h) + b(Y + k) + c}{d(x + h) + p(Y + k) + r}\right) \\ &= F\left(\frac{aX + bY + c + ah + bk + c}{dX + pY + r + dh + pk + r}\right). \end{aligned}$$

This equation is homogeneous exactly when h and k can be chosen so that

$$ah + bk = -c \text{ and } dh + pk = -r.$$

This 2×2 system of algebraic equations has a solution exactly when the determinant of the coefficients is nonzero, and this is the condition that

$$\begin{vmatrix} a & b \\ d & p \end{vmatrix} = ap - bd \neq 0.$$

16. Comparing this with problem 15, we have

$$a = 0, b = 1, c = -3, d = p = 1 \text{ and } r = -1.$$

The system to solve for h and k is

$$k = 3, h + k = 1.$$

Then $k = 3$ and $h = -2$. Let $X = x - 2, Y = y + 3$ to obtain

$$\frac{dY}{dX} = \frac{Y}{X + Y}.$$

This is a homogeneous equation solved in problem 5. The general of the current problem is defined by

$$(y - 3) \ln |y - 3| - (x + 2) = c(y - 3),$$

with c an arbitrary constant.

17. Let $X = x-2, Y = y+3$ to get the homogeneous equation

$$\frac{dY}{dX} = \frac{3X - Y}{X + Y}.$$

The general solution of the original equation (in terms of x and y) is defined by

$$3(x-2)^2 - 2(x-2)(y+3) - (y+3)^2 = c,$$

with c an arbitrary constant.

18. Set $X = x+5, Y = y+1$ to obtain the implicitly defined general solution

$$(x+5)^2 + 4(x+5)(y+1) - (y+1)^2 = c.$$

19. Let $X = x-2, Y = y+1$ to obtain the general solution given by

$$(2x + y - 3)^2 = c(y - x + 3).$$

20. Suppose at time $t = 0$ the dog is at the origin of an $x, y-$ coordinate system, and the person is at $(A, 0)$. The person moves directly upward and at time t is at (A, vt), while the dog is at (x, y) and runs toward the person at a speed $2v$. The tangent to the dog's path joins these two points, and so has slope

$$y' = \frac{vt - y}{A - x}.$$

To find the equation of the dog's path, we will first eliminate t from this equation. In the time the person has moved vt units upward, the dog has run $2vt$ units along its path of motion, so

$$2vt = \int_0^x \left(1 + \left(\frac{dy}{d\xi}\right)^2\right)^{1/2} d\xi.$$

Then

$$vt = y + (A - x)\frac{dy}{dx} = \frac{1}{2}\int_0^x \left(1 + \left(\frac{dy}{d\xi}\right)^2\right) d\xi.$$

Then

$$2(A - x)y' = \int_0^x \left(1 + \left(\frac{dy}{d\xi}\right)^2\right) d\xi - 2y.$$

Differentiate this equation to get

$$2(A - x)y'' - 2y' = \left(1 + \left(\frac{dy}{dx}\right)^2\right) - 2y',$$

so

$$2(A - x)y'' = \left(1 + (y')^2\right)^{1/2},$$

together with the conditions $y(0) = y'(0) = 0$.

Now let $u = y'$ and rewrite the resulting equation to get

$$\frac{1}{(1+u^2)^{1/2}}\,du = \frac{1}{2(A-x)}\,dx.$$

This has the general solution

$$\ln(u + \sqrt{1+u^2}) = -\frac{1}{2}\ln(A-x) + c.$$

Use the condition that $y'(0) = u(0) = 0$ to obtain

$$u + \sqrt{1+u^2} = \left(\frac{A}{A-x}\right)^{1/2}.$$

In terms of y, we now have

$$y' + \sqrt{1+(y')^2} = \frac{\sqrt{A}}{\sqrt{A-x}}; y(0) = 0.$$

But

$$\sqrt{1+(y')^2} = 2(A-x)y'',$$

so

$$y' + 2(A-x)y'' = \frac{\sqrt{A}}{\sqrt{A-x}}.$$

Let $w = y'$ to get

$$w' + \frac{1}{2(A-x)}w = \frac{\sqrt{A}}{2(A-x)^{3/2}}.$$

This linear differential equation has integrating factor $1/\sqrt{A-x}$, so

$$\left(\frac{w}{\sqrt{A-x}}\right)' = \frac{A}{2(A-x)^2}.$$

Integrate this to get

$$w = \frac{\sqrt{A}}{2}\frac{1}{\sqrt{A-x}} + c\sqrt{A-x}.$$

Use the fact that $w(0) = 0$ to get

$$w = \frac{\sqrt{A}}{2}\frac{1}{\sqrt{A-x}} - \frac{1}{2\sqrt{A}}\sqrt{A-x} = y'.$$

Integrate this to get

$$y = -\sqrt{A}\sqrt{A-x} + \frac{1}{3\sqrt{A}}(A-x)^{3/2} + c.$$

Because $y(0) = 0$,

$$y = -\sqrt{A}\sqrt{A-x} + \frac{1}{3\sqrt{A}}(A-x)^{3/2} + \frac{2}{3}A.$$

Now the dog catches the person at $x = A$, so they meet at $(A, 2A/3)$. This is also the point (A, vt), so $vt = 2A/3$ and they meet at time

$$t = \frac{2A}{3v}.$$

21. It is convenient to use polar coordinates to formulate a model for this problem. Put the origin at the submarine at the time of sighting, and the polar axis the line from there to the destroyer at this time (the point $(9, 0)$). Initially the destroyer should steam at speed $2v$ directly toward the origin, until it reaches $(3, 0)$. During this time the submarine, moving at speed v, will have moved three units from the point where it was sighted. Let $\theta = \varphi$ be the ray (half-line) along which the submarine is moving.

Upon reaching $(3, 0)$, the destroyer should execute a search pattern along a path $r = f(\theta)$. The object is to choose this path so that the sub and the destroyer both reach $(f(\varphi, \varphi)$ at the same time T after the sighting.

From sighting to interception, the destroyer travels a distance

$$6 + \int_0^{\varphi} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta,$$

so

$$T = \frac{1}{2v}\left(6 + \int_0^{\varphi} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta\right).$$

For the submarine,

$$T = \frac{1}{v} f(\varphi).$$

Equate these two expressions for T and differentiate with respect to φ to get

$$\frac{1}{2}\sqrt{(f(\varphi))^2 + (f'(\varphi))^2} = f'(\varphi).$$

Denote the variable as θ and rearrange the last equation to obtain

$$\frac{f'(\theta)}{f(\theta)} = \pm\frac{1}{\sqrt{3}}.$$

The positive sign here indicates that the destroyer should execute a starboard (left) turn, while the negative sign is for a portside turn. Taking the positive sign, solve for $f(\theta)$ to get

$$f(\theta) = ke^{\theta/\sqrt{3}}.$$

Now $f(0) = k = 3$, so the path of the destroyer is part of the graph of

$$f(\theta) = 3se^{\theta/\sqrt{3}}.$$

After sailing directly to $(3, 0)$, the destroyer should execute this spiral pattern. A similar conclusion follows if the negative sign of $1/\sqrt{3}$ is used.

This shows that the destroyer can carry out a maneuver that will take it directly over the submarine at some time. However, there is no way to solve for the specific time, so it is unknown when this will occur.

22. (a) Observe that each bug follows the same curve of pursuit relative to the center from which it starts. Place a polar coordinate system as suggested and determine the pursuit curve for the bug starting at $\theta = 0, r = a/\sqrt{2}$. At $t > 0$, the bug will be at $(f(\theta), \theta)$ and its target is at $(f(\theta), \theta + \pi/2)$. Show that

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)cos(\theta) - f(\theta)\sin(\theta)}.$$

At the same time, the direction of the tangent must be from the position $(f(\theta), \theta)$ to the target location $(f(\theta), \theta + \pi/2)$, so we also have

$$\begin{aligned}\frac{dy}{dx} &= \frac{f(\theta)\sin(\theta + \pi/2) - f(\theta)\sin(\theta)}{f(\theta)\cos(\theta + \pi/2) - f(\theta)\cos(\theta)} \\ &= \frac{\cos(\theta) - \sin(\theta)}{-\sin(\theta) - \cos(\theta)} = \frac{\sin(\theta) - \cos(\theta)}{\sin(\theta) + \cos(\theta)}.\end{aligned}$$

Equate these two expressions for dy/dx and rearrange terms to get

$$f'(\theta) + f(\theta) = 0.$$

Further, $f(0) = a/\sqrt{2}$. This is a separable, and also linear, differential equation, and the initial value problem has the solution

$$r = f(\theta) = \frac{a}{\sqrt{2}}e^{-\theta}.$$

This is the pursuit curve (in polar coordinates).

(b) The distance traveled is

$$\begin{aligned}&\int_0^\infty \sqrt{r^2 + (r')^2}\, d\theta \\ &= \int_0^\infty \left[\left(\frac{a}{\sqrt{2}}e^{-\theta}\right)^2 + \left(-\frac{a}{\sqrt{2}}e^{-\theta}\right)^2\right]^{1/2} d\theta \\ &= a\int_0^\infty e^{-\theta}\, d\theta = a.\end{aligned}$$

(c) Because $r = ae^{-\theta}/\sqrt{2} > 0$ for all θ, no bug actually catches its quarry. The actual distance between pursuer and quarry is $ae^{-\theta}$.